Solutions

3.1-3.2: Recurrence Relations and Induction

Question 1. Laura is given a starting salary of \$2,000 and is promised a 10% raise every month. What will her monthly salaries be for the next three months?

Now	1 month	2 months	3 months
\$2,000	\$2,200	\$2,420	\$2,662

Question 2. The *odd numbers* are the numbers in the sequence $1, 3, 5, 7, 9, \ldots$. Define the sequence of *S-numbers* as follows:

The first S-number is 1.
The second S-number is the sum of the first S-number and the second odd number.
The third S-number is the sum of the second S-number and the third odd number.
The fourth S-number is the sum of the third S-number and the fourth odd number, etc., ...

Compute the first seven S-numbers. Make a conjecture.

$S_1 = 1$

$S_2 = 1+3 = 4$

$S_3 = 4+5 = 9$

$S_4 = 9+7 = 16$

$S_5 = 16+9 = 25$

$S_6 = 25+11 = 36$

$S_7 = 36+13 = 49$

$\Rightarrow S_{n+1} = S_n +$ (n+1)st odd number

S_n = Sum of the first n odd numbers.

We are used to defining functions with explicit formulas. For instance, $P : \mathbb{N} \to \mathbb{Z}$ given by

$$P(n) = \frac{n(n+1)}{2}.$$

To the contrary, a <u>recurrence relation</u> is used to define a function *recursively*, that is, each term is defined using terms that were already defined. For instance,

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ n + P(n-1) & \text{if } n > 1. \end{cases}$$

Example 1. Compute $P(5)$ using the above equation.

$P(1) = 1$

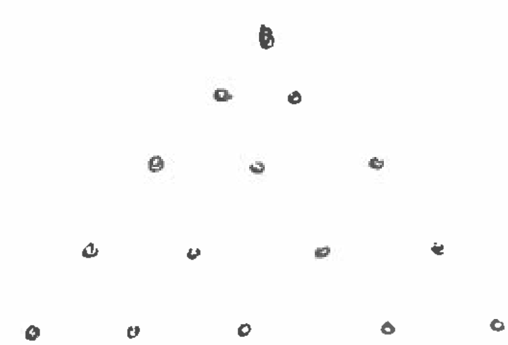
$P(2) = 2+1 = 3$

$P(3) = 3+3 = 6$

$P(4) = 4+6 = 10$

$\boxed{P(5) = 5+10 = 15}$

These are triangle numbers



$P(5)$ = # of dots above

Example 2. Verify that the two equations for P on the previous page are equal using induction.

We want to show that $P(n) = n + P(n-1)$ using the formula $P(n) = \frac{n(n+1)}{2}$.

Base Case: $P(1) = \frac{1\cdot(1+1)}{2} = 1$ ✓

Inductive Step: Suppose $P(n) = \frac{n(n+1)}{2}$.

Then $P(n+1) = n + P(n) = n + \frac{n(n+1)}{2} = \frac{2n + n(n+1)}{2} = \frac{(n+1)(n+2)}{2}$. ✓

So $P(n)$ has the given formula.

Example 3. The Fibonacci Sequence
A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

The Fibonacci numbers $F(n)$ satisfy the following recurrence relation:

$$F(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = 2 \\ F(n-1) + F(n-2) & \text{if } n > 2. \end{cases}$$

Find the first 10 Fibonacci numbers. Is there a closed form for the Fibonacci numbers?

$F(1) = 1 = F(2)$, $F(3) = 1+1 = 2$, $F(4) = 1+2 = 3$,

$F(5) = 2+3 = 5$, $F(6) = 3+5 = 8$, $F(7) = 5+8 = 13$,

$F(8) = 8+13 = 21$, $F(9) = 13+21 = 34$, $F(10) = 21+34 = 55$.

Yes, there is but it is not easy to see.

Fibonacci numbers are seen often in nature whenever growth occurs in stages. See Figure 3.2 on page 155 to see Fibonacci sequences in nature.

Example 4. Ursula the Usurer lends money at outrageous rates of interest. She demands to be paid 10% interest *per week* on a loan, compounded weekly. Suppose you borrow \$500 from Ursula. If you wait four weeks to pay her back, how much will you owe?

$$U(n) = \begin{cases} 500, & n=0 \\ 1.1(U(n-1)), & n>0 \end{cases}$$

$U(1) = 500 \cdot 1.1 = 550$

$U(2) = 550 \cdot 1.1 = 605$ $\quad \Rightarrow \quad U(n) = 500 \cdot (1.1)^n$

$U(3) = 605 \cdot 1.1 = 665.5$

$U(4) = \cancel{7}665.5 \cdot 1.1 = 732.05$

Example 5. Let X be a finite set with n elements. Find a recurrence relation $C(n)$ for the number of elements in the power set $P(X)$. Find a closed form solution and verify it is correct by induction.

$|X| = 0 \Rightarrow |P(X)| = 1$

$|X| = 1 \Rightarrow |P(X)| = 2$

$|X| = 2 \Rightarrow |P(X)| = 4$

$|X| = 3 \Rightarrow |P(X)| = 8$

We can guess that

$C(n) = 2^n$ for all $n \geq 0$.

$$C(n) = \begin{cases} 1, & n=0 \\ 2 \cdot C(n-1), & n>0 \end{cases}$$

Since every subset of X is either a subset of $X' = X \setminus \{x\}$ or of the form $U \cup \{x\}$ for some $U \subseteq X'$.

Induction:

Base Case: $n=0$, $C(0) = 1 = 2^0$ ✓

Inductive Case: Suppose $C(n) = 2^n$ for some $n \geq 0$.

Then $C(n+1) = 2 \cdot C(n) = 2 \cdot 2^n = 2^{n+1}$ ✓

So $C(n) = 2^n$ for all $n \geq 0$.

Example 6. Recall that the complete graph K_n on n vertices is the undirected graph that has exactly one edge between every pair of vertices. Find a recurrence relation $E(n)$ for the number of edges in K_n. Find a closed form solution and verify it is correct by induction.

$$E(n) = \begin{cases} 0, & n=1 \\ E(n-1)+(n-1), & n \geq 2 \end{cases}$$

K_n can be constructed from K_{n-1} by adding a vertex and an edge to it from all $n-1$ vertices of K_{n-1}.

$E(1)=0$, $E(2)=1$, $E(3)=3$, $E(4)=6$, $E(5)=10$, ... See Example 2.

Guess $E(n)=\frac{n(n-1)}{2}$.

Induction: Base Case: $n=1$, $E(1)=0=\frac{1\cdot 0}{2}$ ✓

Inductive Step: Suppose $E(n)=\frac{n(n-1)}{2}$ for some $n\in\mathbb{N}$.

Then $E(n+1)=E(n)+n=\frac{n(n-1)}{2}+n=\frac{n(n-1)+2n}{2}$

$=\frac{n^2+n}{2}=\frac{n(n+1)}{2}$ ✓. So $E(n)=\frac{n(n-1)}{2}$ $\forall n\in\mathbb{N}$

Example 7. Use the *sequence of differences* to find a closed form solution for the recurrence relation

$$H(n) = \begin{cases} 1 & \text{if } n = 1 \\ H(n-1) + 6n - 6 & \text{if } n > 1. \end{cases}$$

$n=$	1	2	3	4	5	6
$H(n)$	1	7	19	37	61	91
$H(n+1)-H(n)$	6	12	18	24	30	
Differences Again	6	6	6	6		

This suggests $H(n)=An^2+Bn+C$ (since the tree has depth 2).

Since $\left.\begin{array}{l} H(1)=1 \\ H(2)=7 \\ H(3)=19 \end{array}\right\} \Rightarrow \begin{array}{l} 1=A+B+C \\ 7=4A+2B+C \\ 19=9A+3B+C \end{array} \Rightarrow \begin{array}{l} A=3 \\ B=-3 \\ C=1 \end{array} \Rightarrow H(n)=3n^2-3n+1$

is a good guess which still needs to be inductively verified.

Practice Problems. Section 3.1: 2, 3, 5, 7, 9-11, 14, 20-23, **26, 27**; Section 3.2: 1-9 odd, 13, **17-20**